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An Approach to Constructing Sets with Imaginary and Complex Elements

Аннотация. When solving equations with sets problems arise that there is no solution for some of them in the framework of the classic Cantor's set theory. Introduction of an imaginary set, different from the real (classical) set is proposed in the present work. At that the union of an element or a set located in a real state with the same element or set from the imaginary state result in an empty state. This means that one and the same element in a given set may be in one only state – real or imaginary but not in both simultaneously. The set in which different elements in one of the two possible states are contained is called a complex set. With regard to these features the classical operations - ∪, ∩, \, Δ, are adapted to complex ones. Some relations are obtained characteristic to the complex sets only. It is shown that these sets observe the algebraic requirements intrinsic to the Boolean algebra and the lattice, De Morgan's laws for double negation, commutativity, distributivity and other. An example is shown for constructing a boolean of the complex sets.

Ключевые слова: sets-real, imaginary, complex; complex sets operations; Boolean algebra; lattice.

Introduction

Different problems had arisen in the centuries old history of mathematics and nowadays – in Information Technologies (IT), which could not be solved in the frame of the structures established. This had imposed extension of these structures. An example for this may be pointed out in the real number theory in which an equation arises in the form:

$$x^2 = -1;\ x = \sqrt{-1}, \tag{1}$$

which cannot be solved in the real numbers' frame. No real number exists which raised to square results in -1. This had imposed the extensions of real numbers' scope to the complex numbers [4] thorough the imaginary unit i ($i^2 = -1$). The general well-known form of these complex numbers z is:

$$z = a + bi, \tag{2}$$

where a and b are real numbers.

Analogically in [7] an imaginary logical variable i was introduced accepting two states i and $\neg i$. This provided a possibility of solving logical equations of the form of $F \wedge x = T$ through the relation

$$F \wedge i = T \tag{3}$$

where $\{F, T\}$ are the states of the classical real (r-) propositional logic, and $\{i, \neg i\}$ is the set of states in the imaginary logic.

A solution of the set equation [2, 3, 5]

$$A \cup B = C; \tag{4}$$

is sought in the present work, where Ø is the empty set, and the symbols ∪, ∩, \, Δ denote the respective operations union, intersection, difference, and symmetric difference in the Cantor's set theory [2], and A, B, and C are respective subsets. If we suppose that in equation (4) non-empty sets A and B are given, such that

$$A \backslash C \neq \emptyset \text{ and } C \backslash A \neq \emptyset. \tag{5}$$

then, as seen from Fig. 1 no such a set B exists in the frame of the classic set theory that satisfies the requirements of (4).

This impossibility is still more obvious if we suppose in equation (4)

$$A \neq \emptyset;\ C \neq \emptyset; \tag{6}$$

because there is no way that the non-empty set A united with any other set B to result in an empty set $C = \emptyset$.

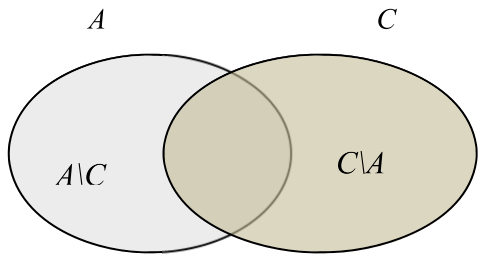

Fig. 1

1. Real and imaginary sets

An outcome of the situation may be found if we suppose that every element $a \in U$, where U is the universal set may be in one of two its sets – real a and imaginary – ia, where i – a symbol of imaginarity. Then another point of view may exist, where those two states of one and the same element a may be considered as two distinct elements

$$a \in U, \text{ and } ia \in \{ia/a \in U\}. \qquad (7)$$

We assume that new complex elements may be formed from (7) by a union between a and ib, namely:

$$c = a \cup ib;\ a \in U;\ ib \in \{ib/b \in U\}; \qquad (8)$$

at which $a \neq b$.

Then the universal set

$$U' = \{(a \cup ib) \,/\, a \in U;\ b \in U;\ a \neq b\} \qquad (9)$$

should be considered as a complex universal set in which together and separately three types of elements are contained:

$$\{a\};\ \{ib\};\ \{(a \cup ib)\}. \qquad (10)$$

As a mapping exists between U and U' - and that follows from (9), further both will be denoted by one and the same symbol – U.

For finite sets A, B, and C relations (10) will be in the following form

$$a \in A;\ b \in B;\ (A \cup iB) \subseteq C. \qquad (11)$$

A requirement is introduced that for each $a \in U$ and $ia \in iU$ the following relation is observed:

$$a \cup ia = \emptyset. \qquad (12)$$

This means that condition (12) will be also observed for any finite set $A \subset U$; and $iA \subset iU$, namely:

$$A \cup iA = \emptyset. \qquad (13)$$

With relations from (7) to (13) thus defined a general solution B may be found of the equation (4) and (5) by using the imaginary element and sets:

$$B = C \cup i(A \setminus C);$$
$$\text{i.e. } A \cup B = A \cup (C \cup i(A \setminus C)). \qquad (14)$$

In the case (6), when $C = \emptyset$ the solution is simpler and evident.

$$B = iA;\ A \cup B = A \cup iA = \emptyset. \qquad (15)$$

In this way the class of the solvable set equations is extended through the imaginary elements and sets.

Requirement (12) means that that any element a of one and the same complex set C cannot simultaneously be in two states – real a and imaginary ia, i.e.

$$\text{if } a \in C \text{ then } ia \notin C, \text{ and vice versa.} \qquad (16)$$

Definition: Two complex sets

$$C_1 = A_1 \cup iB_1;\ C_2 = A_2 \cup iB_2; \qquad (17)$$

are considered to be equal if each element $a \in A_1$ is simultaneously an element of A_2 and vice versa and each element of iB_1 is simultaneously an element of iB_2 and vice versa. Sets C_1 and C_2 from (17) are thoroughly defined by their elements. If

$$A = \{a_1, a_2, .., a_n\};\ iB = \{ib_1, ib_2, .., ib_n\};$$
$$A \cap B \neq \emptyset; \qquad (18)$$

then the set $C = A \cup iB$ will be called (n, m) element set.

Set C_1 of (17) is a subset of C_2 if each element of A_2 is an element of A_1 and each element of iB_2 is an element of iB_1. Then we may write down:

$$A_1 \subseteq A_2;\ iB_1 \subseteq iB_2;\ (A_1 \cup iB_1) \subseteq (A_2 \cup iB_2);$$
$$\text{and } C_1 \subseteq C_2. \qquad (19)$$

Another definition may be given: set C_1 is equal to C_2 if

$$((C_1 \subseteq C_2) \text{ and } ((C_2 \subseteq C_1)). \qquad (20)$$

The proof method of set theory equations is known as “method od two inclusions”. In case that $A_1 \cup iB_2 = \emptyset$ and $C_1 \neq C_2$ we write down $C_1 \subset C_2$ and the set C_1 is called a “proper set”.

For the complex sets’ operations $\cup$, $\cap$, \, Δ will be defined observing the requirements:
for each $C = A \cup iB$,

$$A \cap B = \emptyset \text{ and } A \cap iB = \emptyset. \qquad (21)$$

Further on the expedience of such an assumption will be proved through relations (29) to (33). For each C_1 and C_2 from (17):

a) The union of $C_1 \cup C_2$ is equal to:

$$(A_1 \cup iB_1) \cup (A_2 \cup iB_2) = ((A_1 \cup A_2) \cup i(B_1 \cup B_2) =$$
$$= \{a/(a \in A_1 \vee a \in A_2)\} \cup \{(ib/(ib \in iB_1 \vee ib \in iB_2)\}; \qquad (22)$$

where $\vee$ and $\wedge$ are the symbols of disjunction and conjunction respectively. The results of (22) follow from the associativity of the union.

b) The intersection $C_1 \cap C_2$ is equal to:

$$(A_1 \cup iB_1) \cap (A_2 \cup iB_2) = ((A_1 \cap A_2) \cup i(B_1 \cap B_2) = \\ = \{a/(a \in A_1 \wedge a \in A_2)\} \cup \{(ib/(ib \in iB_1 \wedge ib \in iB_2)\}; \tag{23}$$

The relation above immediately follows from:
$(A_1 \cup iB_1) \cap (A_2 \cup iB_2) = (A_1 \cap A_2) \cup (A_1 \cup iB_2) \cup (A_2 \cap iB_1) \cup i(B_1 \cap B_2) = (A_1 \cap A_2) \cup i(B_1 \cap B_2)$;
due to the fact that by assumption (21)
$A_1 \cap iB_2 = Ø$ and $A_2 \cap iB_1 = Ø$.

c) The difference $C_1 \setminus C_2$ is equal to

$$(A_1 \cup iB_1) \setminus (A_2 \cup iB_2) = (A_1 \setminus A_2) \cup i(B_1 \setminus B_2) = \\ =\{a/(a \in A_1 \wedge a \notin A_2)\}\cup\{ib / (ib \in iB_1 \wedge ib \notin iB_2)\}. \tag{24}$$

This follows from:

$$(A_1 \cup iB_1) \setminus (A_2 \cup iB_2) = \\ = ((A_1 \setminus (A_2 \cup iB_2)) \cup ((iB_1 \setminus (A_2 \cup iB_2)) = \\ = ((A_1 \setminus A_2) \cap (A_1 \setminus iB_2)) \cap ((iB_1 \setminus A_2) \cup (A_1 \cup iB_1) \\ \cup ((iB_1 \setminus A_2) \cap ((iB_1 \setminus A_2) \cap (iB_1 \setminus iB_2)) = \\ ((A_1 \setminus A_2) \cap A_1)) \cup ((iB_1 \cap i(B_1 \setminus iB_2)) = \\ = (A_1 \setminus A_2) \cup i(B_1 \setminus iB_2).$$

In analogic manner:
$(A_2 \cup iB_2) \setminus (A_1 \cup iB_1) = (A_2 \setminus A_1) \cup i(B_2 \setminus B_1)$.

d) Symmetric difference may be defined by the difference (24)

$$C_1 \Delta C_2 = (A_1 \cup iB_1) \Delta (A_2 \cup iB_2) = \\ ((A_1 \cup iB_1) \setminus (A_2 \cup iB_2) \cup ((A_2 \cup iB_2) \setminus (A_1 \cup iB_1)) = \\ (A_1 \setminus A_2) \cup i(B_2 \setminus B_1) \cup (A_2 \setminus A_1) \cup i(B_2 \setminus iB_1) = \\ ((A_1 \setminus A_2) \cup (A_2 \setminus A_1)) \cup i(B_1 \setminus B_2) \cup (B_2 \setminus B_1). \tag{25}$$

e) Splitting is an operation which in the complex sets is defined through the following relations below.

Let sets C_1 and C_2 from (17) be defined for which it is true:

$$C = C_1 \cup C_2 = A \cup iB; \tag{26}$$

and:

$$1.\ A_1 \cup A_2 = A;\ iB_1 \cup iB_2 = iB; \tag{27}$$

$$2.\ A_1 \cap A_2 = Ø;\ iB_1 \cap iB_2 = Ø; \tag{28}$$

Then we may state that the complex set C is split into two subsets C_1 and C_2. Operations $\cup$, $\cap$, $\setminus$, Δ, and splitting are shown in Figs. 2 to 6 with the help of the filled areas.

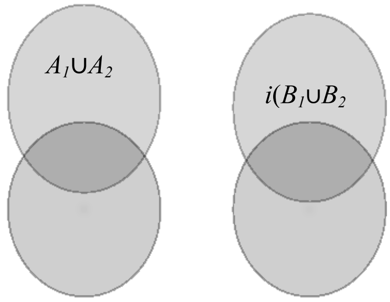

Fig. 2 - ∪

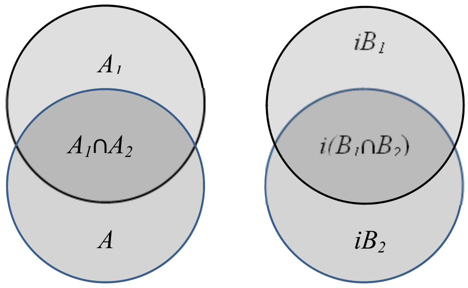

Fig. 3 - ∩

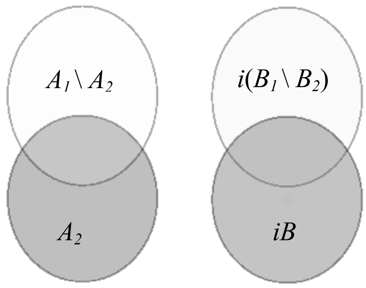

Fig. 4 - \

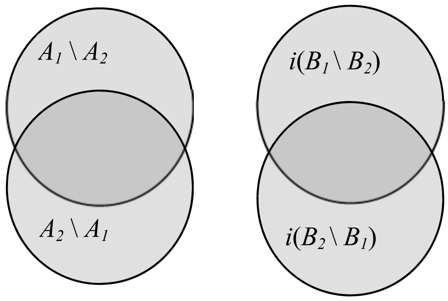

Fig. 5 - Δ

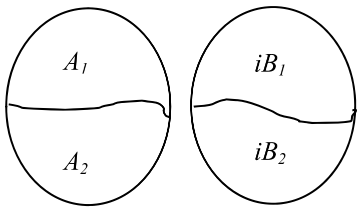

Fig. 6 - splitting

Definition: The complex set $C = A \cup iB$ will be called ***normed*** if it satisfies the condition

$$A \cap B = \emptyset. \quad (29)$$

Proposition: Condition (29) is necessary and sufficient for $C = A \cup iB$ to satisfy the requirements (13) and (14). Violation of condition (29) means that

$$A \cap B = D \neq \emptyset \quad (30)$$

an element, although single, exists which is in real state and simultaneously – in imaginary state $ia \in$ D. But according to (12) and (16) this results in contradiction and proves the the impossibility of (30) and the truth of (29). And vice versa – if (13)and(14) are observed then this results in $D = \emptyset$ and as so to (29).

Hence the complex set $C = A \cup iB$ is normed if it contains one and the same element in two states or which is the same – two elements a and ia, i.e. requirements (29) and (12), (13) are observed together or separately.

Corollary: The following relations are equivalent to (29):

$$\text{a) } A \setminus B = A \text{ and } B \setminus A = B; \quad (31)$$

$$\text{b) } A \cup B = A \,\Delta\, B. \quad (32)$$

Condition (29) is supposed to be observed. Then

a) $A = (A\setminus B) \cup (A\cap B) = (A\setminus B)$
and $B = (B\setminus A) \cup (A\cap B) = (B\setminus A)$;

b) $A\cup B = (A\setminus B) \cup (A\cap B) \cup (B\setminus A) =$
$=(A\setminus B) \cup (B\setminus A) = A \,\Delta\, B$.

The converse is also true – (29) follows from the truth of (31) and (32). This means that if at least one of the equalities (31) and (32) is observed the complex set ($A \cup iB$) is normed and relation (29), and (13), (14) are also observed.

The following relation is almost obvious: for each $A \subset U$ and $iB \subset iU$ it is true that

$$A \cap iB = \emptyset. \quad (33)$$

The assumption that $A \cap B = D \neq \emptyset$ means that at least one element exists which is real and imaginary state simultaneously which is impossible by definition and proves (33).

Proposition: Condition (33) is not sufficient for the normalization of the complex set ($A\cup iB$).
We assume that

$$A = (A^I \cup a);\ B = (B^I \cup a). \quad (34)$$

Then according to (33) and (34) the result $(A \cap iB) = (A^I \cup a) \cap (B^I \cup a) = \emptyset$. The opposite assumption means that one and the same element is simultaneously in both states – real and imaginary, which is impossible.

On the other hand under the same assumption (34)

$$A \cap B = (A^I \cup a) \cap (B^I \cap a) = (A^I \cap B^I) \cup a \neq \emptyset$$

which does not agree with (33) and means that $A \cup iB$ is not normed, that proves the proposition.

Corollary: On the base of the result above it may be proved that the following conditions

$$A \setminus I = A;\ iB \setminus A = iB; \quad (35)$$

$$\text{and } A \cup iB = A \,\Delta\, B; \quad (36)$$

are not sufficient for normalization 0f complex sets and for observing (12) and (13). The proofs for the insufficiency of (35) and (36) may be carried out analogically to those for (31) and (32).

It follows from this that although the norming condition is embedded if the definitions of operations ∪, ∩, \, Δ each operation "union of real and imaginary sets" a check should be carried out for the normalization of the new complex set attained through any of procedures (29), (31), or (32).

Results (33), (35), and (36) show that operations ∪, \, Δ in the complex sets result in either real A or imaginary iB only, or in complex sets $A \cup iB$. This is the cause operation ∪ to be chosen as a basic one in defining the complex set $(A \cup iB) \in U$.

For each complex set C a set of all subsets of C may be formed. It is called *boolean* of set C and it is denoted as 2^C.

$$2^C = \{X / X \subseteq C\}.$$

If A, B, and C are finite sets then it is expedient boolean 2^C to be considered as an aggregate of the real set 2^A boolean, the imaginary set 2^B and the complex set $2^{C'}$ for which
$C' = \{(a \cup ib) / (a \in A \wedge b \in iB); A \neq \emptyset; B \neq \emptyset; a \neq b\}$
then $2^C = 2^A \cup 2^B \cup 2^{C'}$.

If we denote by $|A|$ the number of elements in A then

$$2^{|C|} = 2^{|A|} + 2^{|B|} + 2^{|C'|}.$$

Let $A = \{a_1, a_2\}$ and $iB = \{ib_1, ib_2, ib_3,\}$,
from which it follows $|A| = n = 2$; $|B| = m = 3$; $|C| = n.m = 6$.
Then $2^{|C|} = 2^n + 2^m + 2^{nm} = 2^{|C|} = 2^2 + 2^3 + 2^6 = 76$.

This means that 76 subset correspond to the complex set $\{a_1, a_2\} \cup \{ib_1, ib_2, ib_3,\} = A \cup iB$ – much more than th real set – 4 and the imaginary set iB – 8. Increasing number of complex sets may be generated by the boolean 2^C.

Complements of the sets A, B, and C respectively will be defined through the universal set U in the following way:

$$\bar{A} = U \backslash A;\ \bar{B} = U \backslash B;\ \bar{C} = U \backslash C. \quad (37)$$

These complements are sets of all elements of U not belonging to A, B, and C respectively. Further besides the denotations (17) for C_1 and C_2 the denotation $C_3 = A_3 \cup iB_3$.

It is well known that Cantor's set theory has algebraic features characteristic to the Boolean algebra and the lattice. In the case under consideration such features have separately both the real and the imaginary sets. It will be demonstrated that the complex sets are constructed in such a manner that they also have such features. These algebraic features are demonstrated in the following Table 1 with regard to the complex sets. Requirements (13) and (36) are shown in it, characteristic to the complex sets only.

It follows from the way of defining of operations over the complex sets (22) to (25), as well as from the remaining relations from (19) to (36) that for the complex sets the algebraic laws: associative; communicative, idempotent, distributivity of the union with regard to intersection and vice versa, are also observed.

It will be shown that the two De Morgan's laws (rules 7 and 8 from Table 1) are also observed. The complements will be defined:

$$\overline{C_1} = U \backslash C_1;\ \overline{C_2} = U \backslash C_2;$$

and $\overline{C_1 \cup C_2} = \mathrm{U} \backslash (C_1 \cup C_2) = (\mathrm{U} \backslash C_1) \cap (\mathrm{U} \backslash (C_2)$.

The comparison between the three relations above confirms the first De Morgan's law (rule 7 of Table 1). The second De Morgan's law may be confirmed in analogic way by the next rule 8. It may be shown that the law of double complement from rule 17 is used in the complex sets also:

$$\overline{C_1} = U \backslash C_1;$$

$$\overline{\overline{C_1}} = U \setminus (U \setminus C_1) = (U \setminus U) \cup (U \cap C_1) = C_1.$$

The truth of the remaining rules from Table 1 may be proven in a similar way and this is comparatively not complicated. As a rule keeping of the same rules in the complex logic like in the classical Cantor's set theory is a corollary of the fact, that operations $\cup$, $\cap$, \, Δ from (22) to (25) for the complex sets are as a rule executed separately for the real and imaginary sets and in the framework of the classical rules. The real and imaginary sets practically do not intersect in between and they interact mainly through requirements (13) and (36), rules (13) and (36) respectively from Table 1. Due to this cause mainly in the complex sets themselves very complicated cases do not arise. The similarity of the complex sets to algebraic structures Boolean algebra and lattice emerges from Table 1.

The fact is to be noted that the imaginary set play the role of a negative set. This immediately follows from the initial assumption (13) according to which the "collision" at the uniting the real set A with the imaginary set iA results in an empty set. This means that the term "negative set" may be used in this sense. The term "imaginary set" is preferable though because the role it plays is in essence the same like the imaginary part of the complex numbers.

Another point of view may be formulated where the real and imaginary sets are considered as two specific subsets between which interactions (13) and (14) exist in the framework of the classical Cantor's theory. Anyway they provide additional specific possibilities and extend the possibilities of the classical logic to some degree.

Table 1

1.	$C_1 \cup C_2 = C_2 \cup C_1$	13.	$C_1 \cup \overline{C_1} = U$
2.	$C_1 \cap C_2 = C_2 \cap C_1$	14.	$C_1 \cap \overline{C_1} = \emptyset$
3.	$C_1 \cup (C_2 \cup C_3) = (C_1 \cup C_2) \cup C_3$	15.	$C_1 \cup C_1 = C_1$
4.	$C_1 \cap (C_2 \cap C_3) = (C_1 \cap C_2) \cap C_3$	16.	$C_1 \cap C_1 = C_1$
5.	$C_1 \cap (C_2 \cup C_3) = (C_1 \cap C_2) \cup (C_1 \cap C_3)$	17.	$\overline{\overline{C_1}} = C_1$
6.	$C_1 \cup (C_2 \cap C_3) = (C_1 \cup C_2) \cap (C_1 \cup C_3)$	18.	$C_1 \backslash C_2 = C_1 \cap \overline{C_2}$
7.	$\overline{C1 \cup C2} = \overline{C1} \cap \overline{C2}$	19.	$C_1 \Delta C_2 = (C_1 \cup C_2) \setminus (C_1 \cap C_2)$
8.	$\overline{C1 \cap C2} = C_1 \cup C_2$	20.	$(C_1 \Delta C_2) \Delta C_3 = C_1 \Delta (C_2 \Delta C_3)$
9.	$C_1 \cup \emptyset = C_1$	21.	$C_1 \Delta C_2 = C_2 \Delta C_1$
10	$C_1 \cap \emptyset = \emptyset$	22.	$C_1 \cap (C_2 \Delta C_3) = (C_1 \cap C_2) \Delta (C_1 \cap C_3)$
11.	$C_1 \cap U = C_1$	23.	$A \cup iA = \emptyset$
12	$C_1 \cup U = U$	24.	$C_1 \Delta C_2 = C_1 \cup C_2$

2. Summing up

A series of questions arises concerning the possibilities of using the complex sets:

a) As the set theory is basic for constructing of various mathematical structures then it is possible for some of them new possibilities for extension to arise;

b) The example cited – from (12) to (15) demonstrates how by introducing of complex sets various set equations may be solved;

c) On the base of the complex sets proposed, construction of physical models is possible in which the union of different elements (particles) of different, opposite charge results in their mutual destruction (annihilation) or transition to another state.[1]

The complex sets being proposed need wider, deeper, and more rigorous investigation which may lead to better definition of their possibilities.

[1] In general, the union operation does not satisfy the associativity law, hence these operations must be delayed to the end stage of computation.

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