Solving Problems in Transportation Systems Modeled by the Nonlinear Kolmogorov-Feller Equation

A. N. Firsov[I], A. A. Zhilenkov[II], S. G. Chernyi[III, IV]

[I] Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russia
[II] St. Petersburg State Marine Technical University, Saint Petersburg, Russia
[III] Kerch State Maritime Technological University, Kerch, Russia
[IV] Admiral Makarov State University of Maritime and Inland Shipping, Saint Petersburg, Russia

Abstract. The paper describes the construction of a solution to the Kolmogorov-Feller equation with a nonlinear drift coefficient. This and similar equations are used in problems of the theory of transport and diffusion. Equations of this type are found in stochastic problems of the theory of safety and reliability, the dynamics of stellar systems, and even in economic problems. The paper proposes a constructive method for solving the stationary Kolmogorov-Feller equation with a nonlinear drift coefficient. The corresponding algorithms are constructed and their convergence is justified. The basis of the proposed method is the application of the Fourier transform.

Keywords: Kolmogorov-Feller equation; nonlinear drift coefficient; constructive method for solving.

DOI 10.14357/20718632210209

Introduction

The problems of developing and analyzing mathematical models of physical transfer and filtration processes in technical objects can be studied in terms of the correctness of problems for differential, integral and integro-differential equations. Numerical methods for the analysis of such problems are formed on the basis of converging algorithms. The best condition is to found analytical solutions to problems in the form of recurrence relations justifying their convergence and assessing the accuracy of approximations. Decomposition of the problem in the study of equations, and its solution through a set of engineering and technical problems allows using integral transforms of the Fourier or Laplace type or formal decompositions of the sought solutions into function series. However, these methods are not always effective, since the application of the Fourier or Laplace transforms for nonlinear operators that transform themselves does not simplify the problem. Moreover, decomposition methods of the sought functions into series contain coefficients that do not always have technical or physical basis. Finally, when studying technical and physical problems using the indicated methods, the problem of inverting the corresponding transformations arises.

Thus, it is of great interest to develop methods for solving differential, integral and integro-differential equations that describe transfer and filtration phenomena in technical and physical problems. The solution of these problems through the above-mentioned methods has only advantages. It also provides an opportunity for analytical research and the construction of analytical modeling methods and effective algorithms for the approximate solution of further technical and physical problems using models of mathematical physics, theory of random processes, control theory, etc. The author's research shows that the development and substantiation of modeling methods and solving applied problems is possible using the theory of generalized functions. The thesis is devoted to the development, substantiation and applications of the proposed method for solving technical problems of mathematical physics, describing the processes of transfer and filtration, which decay rather quickly at infinity. The method is based on the construction and analysis of a new class of generalized functions as linear functionals in spaces of entire functions of many real variables. In particular, a constructive connection is traced between the indicated functions and the sequence of their "power moments", which makes it possible to give a complete and constructive solution to the problem of moments for the functions of the classes under consideration.

1. Mathematical Model of the Problem

Consider a variant of the Kolmogorov-Feller equation (1), which occurs in control theory, communication theory, stellar dynamics. In the literature devoted to analytic constructions of solutions of equations of type (1), the case of a linear dependence ($\beta = 0$) of the drift coefficient on the coordinate is usually considered; see, for example [1, 2]. In this paper, we will consider $\beta \neq 0$.

$$\frac{d}{dx}[(\alpha x+\beta x^2)W(x)]+\nu\int_{-\infty}^{+\infty}p(A)\ \ W(x-A)dA-\nu W(x)=0,-\infty<x<+\infty \quad (1)$$

under natural conditions

$$W(x)\underset{x\to\pm\infty}{\to 0},\qquad \int_{-\infty}^{+\infty}W(x)dx=1 \quad (2)$$

$$p(A)\xrightarrow[|A|\to\infty]{}0,\quad \int_{-\infty}^{+\infty}p(A)\ \ dA=1. \quad (3)$$

Additionally, suppose $p(A)$ - analytical function with $|A|<R$ for R big enough or at least its Fourier transform $\hat{p}(k)=\int_{-\infty}^{+\infty}p(x)\ \ e^{ixk}dx$ exists and is an analytical function in a sufficiently large interval [3]:

$$\hat{p}(k)=\hat{p}_0+\hat{p}_1k+\hat{p}_2k+\cdots,|k|<\ k_0,\qquad k_0\gg 1 \quad (4)$$

Note that

$$\hat{p}_s=\frac{\hat{p}^{(s)}(0)}{s!}=\frac{1}{s!}(i)^s\int_{-\infty}^{+\infty}x^s p(x)\ \ dx \quad (5)$$

In particular, by virtue of (3),

$$\hat{p}_0=\hat{p}(0)=1 \quad (6)$$

Also, if $p(x)$ - even function, then$\hat{p}_{2s-1}=0,\quad s=1,2,\dots$, and $\hat{p}(k)$ - is real.

We now turn in equation (1) to the Fourier transform of the function $W(x)$:

$$\widehat{W}(k)=\int_{-\infty}^{+\infty}W(x)e^{ixk}dx,$$

$$k\left[i\beta\widehat{W}''-\alpha\widehat{W}'+\nu\left(\frac{\hat{p}-1}{k}\right)\widehat{W}\right]=0.$$

Conditions (2) at the same time go to

$$\begin{cases} \int_{-\infty}^{+\infty} |\widehat{W}(k)| dk < \infty, \\ \widehat{W}(0) = 1. \end{cases} \tag{7}$$

Thus, the problem of solving an equation (1) with conditions (2) can be replaced by a solution of an equation [3, 4]:

$$i\beta\widehat{W}''(k) - \alpha\widehat{W}'(k) + \nu\rho(k)\widehat{W}(k) = 0, \tag{8}$$

for $\widehat{W}(k)$ satisfying conditions (7). Denoted here

$$\rho(k) = \frac{\hat{p}(k) - 1}{k} \tag{9}$$

By virtue of (6), and the conditions imposed on $\hat{p}(k)$, we have

$$\rho(0) = \hat{p}_1 = \int_{-\infty}^{+\infty} xp(x)\, dx \tag{4}$$

$$p(k) = \hat{p}_1 + \hat{p}_2 k + \hat{p}_3 k^2 + \cdots, |k| < k_0. \tag{5}$$

Moreover [4,5,6] since $\hat{p}(k) \to 0,\ |k| \to \infty$ (because $\hat{p}$ is a Fourier transform), then

$$p(k) \sim \frac{-1}{k} \tag{12}$$

2. Mathematical Model Analysis

Put:

$$\widehat{W}(k) = \varphi(k)e^{-\int_0^k \psi(k)dk}. \tag{13}$$

Notice, that

$$\varphi(0) = \widehat{W}(0) = 1 \tag{14}$$

and

$$\varphi(-k) = \overline{\varphi(k)} \tag{15}$$

(The last equality is a consequence of the choice of $\psi(k)$, which will be done below – see (16)). Substituting (12) into (8), we get

$$\varphi'' + \left(-2\psi + i\frac{\alpha}{\beta}\right)\varphi' + \left(\psi^2 - \psi' - i\frac{\alpha}{\beta}\psi - i\frac{\nu}{\beta}\rho\right)\varphi = 0.$$

Putting on here

$$\psi = i\frac{\alpha}{2\beta}, \tag{16}$$

We'll get for$\varphi(k)$ following equation

$$\varphi'' - q(k)\varphi = 0, \tag{17}$$

Where

$$\varphi(k) = \widehat{W}(k)e^{i\frac{\alpha}{2\beta}k}, \tag{18}$$

$$q(k) = -\frac{\alpha^2}{2\beta^2} + i\frac{\nu}{\beta}\rho(k). \tag{19}$$

Let us note some elementary properties of $q(k)$.

1) By virtue of (12)

$$q(k)+\frac{\alpha^2}{2\beta^2}\sim -i\frac{\nu}{\beta k}\quad (|k|\to\infty) \tag{20}$$

and in particular,

$$\hat{p}(k)\to\frac{-\alpha^2}{2\beta^2},\ (|k|\to\infty) \tag{21}$$

2) By virtue of (11)

$$q(k)=-\frac{\alpha^2}{2\beta^2}+i\frac{\nu}{\beta}\hat{p}_1+i\frac{\nu}{\beta}(\hat{p}_2k+\hat{p}_3k^2+\dots),\quad |k|<k_0 \tag{22}$$

or

$$q(k)=q_0+q_1k+q_2k^2+\cdots,|k|<\ k_0. \tag{22$'$}$$

where

$$\begin{cases} q_0=-\frac{\alpha^2}{2\beta^2}+i\frac{\nu}{\beta}\hat{p}_1,\\ q_n=i\frac{\nu}{\beta}\hat{p}_{n+1}.\end{cases} \tag{22$''$}$$

3) Let

$$\delta=\delta(k)+\frac{\alpha^2}{2\beta^2}+\frac{\nu}{\beta}Im\,\rho\,(k). \tag{23}$$

Then

$$q(k)=-\delta(k)+i\frac{\nu}{\beta}Re\,\rho\,(k). \tag{24}$$

At the same time:
if $\rho(k)$ is real (which will take place, if, for example,$p(x)$ is an even function), then

$$\delta=const=\frac{\alpha^2}{2\beta^2}>0; \tag{25}$$

if $Im\,\rho\,(k)\neq 0$, then by (12), for sufficiently large $|k|$

$$\delta(k)=\frac{\alpha^2}{4\beta^2}+\frac{\nu}{\beta}Im\,\rho\,(k)>0 \tag{26}$$

Lemma 1. The branch of $\sqrt{q(k)}$:

$$\left(\sqrt{q(k)}\right)_1=\sqrt{|q(k)|}\left\{\frac{1}{2}\left(1+\left(1+\frac{[Re\,\rho(k)]^2\nu^2}{\delta^2\beta^2}\right)^{-\frac{1}{2}}\right)\right\}^{\frac{1}{2}}--i\sqrt{|q(k)|}\left\{\frac{1}{2}\left(1-\left(1+\frac{[Re\,\rho(k)]^2\nu^2}{\delta^2\beta^2}\right)^{-\frac{1}{2}}\right)\right\}^{\frac{1}{2}}$$

is twice continuously differentiable by $k\in(0,+\infty)$ and $Re\left(\sqrt{q(k)}\right)_1>0$ for k, which are large enough.

Remarks

1) $\sqrt{|q(k)|}=\left(\delta^2+\frac{\nu^2}{\beta^2}[Re\,\rho\,(k)]^2\right)^{\frac{1}{4}}$

2) If $\rho(k)$ - is real, then $Re\left(\sqrt{q(k)}\right)_1>0$ for all $k>0$.

3) By virtue of property (15), it suffices to construct a solution to equation (17) only for $k\geq 0$.

3. Construction of the Solution of the Transfer Theory Problem

We will use the well-known asymptotic theorem for solving the equation

$$u''(x)-q(x)u(x)=0 \tag{28}$$

when $x \to +\infty$.

Theorem 1. Let in the equation (28) $q(x) \epsilon C^2(0,\infty)$ for sufficiently large x and let there exist a branch $\sqrt{q(x)}$ of class $C^2(b,\infty)$ such that $\sqrt{q(x)} > 0,\ x > b \geq 0$Let further $\alpha_1(x) = \frac{1}{8}\frac{q''}{q^{\frac{3}{2}}} - \frac{s}{32}\frac{[q']^2}{q^{\frac{5}{2}}}$ and. $\int^x |\alpha_1(x)| dx$ Then equation (28) has a solution

$$u(x) = q^{-\frac{1}{4}}(x) e^{-\int^x \sqrt{q(t)}dt}[1 + \varepsilon_2(x)], \quad \varepsilon_2(x) \to 0,\ (x \to \infty)$$

Moreover, for $x > 0$

$$\left|\frac{u(x)}{\tilde{u}(x)} - 1\right| \leq 2\left(e^{2\int_x^\infty |\alpha_1(t)|dt} - 1\right),$$

where $\tilde{u}(x) = q^{-\frac{1}{4}}(x)\, exp\left(-\int^x \sqrt{q(t)}dt\right)$,

$$\left|\frac{u'(x)}{\sqrt{q(x)}\tilde{u}(x)} + 1\right| \leq \frac{1}{4}\left|\frac{q'(x)}{q^{\frac{3}{2}}(x)}\right| + 4\left(1 + \frac{1}{4}\left|\frac{q'(x)}{q^{\frac{3}{2}}(x)}\right|\right) \times \left(e^{2\int_x^\infty |\alpha_1(t)|dt} - 1\right).$$

If $\frac{q'(x)}{q^{\frac{3}{2}}(x)} \to 0,\ (x \to \infty)$ then $u'(x) = q^{\frac{1}{4}}(x) e^{-\int^x \sqrt{q(t)}dt}\left(1 + \varepsilon_1(x)\right)$, $\varepsilon_1(x) \to 0$, $x \to +\infty$.

Lemma 2. If $|\hat{p}(k)| \leq O\left(\frac{1}{k}\right)$ and $\left|\hat{p}''(k)\right| \leq O\left(\frac{1}{k}\right)$ then for equation (17) the previous theorem is valid.

Thus, further we solve the following problem:

$$\varphi''(k) - q(k)\varphi(k) = 0, \quad k > 0 \tag{29}$$

$$\begin{cases} \varphi_0 = 1 \\ \varphi(k) \to 0, \quad k \to +\infty \end{cases} \tag{30}$$

Here $q(k)$ is given by formula (19). Further, we assume that the assumptions of Theorem 1 and all the relations of §2 are fulfilled. In particular, the function $q(k)$ is analytic when $|k| < k_0,\ k_0 \gg 1$ (see (22')).

From the theory of differential equations, it is known that the solutions of equation (29) are analytic in the same domain as $q(k)$. Therefore, the decision $\varphi(k)$ should look like

$$\varphi(k) = 1 + a_1 k + a_2 k + \cdots, \quad 0 \leq k < k_0 \tag{31}$$

If we now substitute this decomposition into (29) and take into account ($22'$), we obtain for the coefficients a_n following infinite system of equations [3,5,7]:

$$\begin{cases} (n+1)(n+2)a_{n+2} - \sum_{s=0}^n a_s q_{n-s} = 0 \\ a_0 = 1 \end{cases} \tag{32}$$

For a_2 we immediately get at $n = 0$

$$a_2 = \frac{1}{2}q_0 = -\frac{\alpha^2}{4\beta^2} + i\frac{\nu}{2\beta}\hat{p}_1. \tag{33}$$

In case of even $p(x)$: $\hat{p}_1 = 0,\quad a_2 = -\frac{\alpha^2}{4\beta^2}$.

Note that if you count known a_1, then the system (32) for $n = 1,2,\ldots,N$ will be a closed system (Nequations for N unknown $a_3,\ldots,a_{N+2}$).

The matrix A_N of this system is

$$A_N = \begin{pmatrix} 2\cdot 3 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 3\cdot 4 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -q_0 & 0 & 4\cdot 5 & 0 & \cdot & \cdot & 0 & 0 \\ -q_1 & -q_0 & 0 & 5\cdot 6 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q_{N-3} & -q_{N-4} & -q_{N-5} & -q_{N-6} & \cdot & -q_0 & 0 & (N+1)(N+2) \end{pmatrix}.$$

Its determinant

$$\triangle_N = det\, A_N = (2\cdot 3)(3\cdot 4)\dots(N+1)(N+2) = \frac{1}{2}(N+1)!\,(N+2)! = \frac{N+2}{2}[(N+1)!]^2 > 0.$$

Therefore, there is an inverse matrix A_N^{-1}, which determinant

$$\triangle_N' = det\, A_N^{-1} = \frac{2}{(N+2)}[(N+1)!]^{-2} \to_{N\to\infty} 0. \qquad (35)$$

Denote further

$$Q_N = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}, \quad Q_N' = \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{N-1} \end{pmatrix}, \quad Q_N'' = \begin{pmatrix} 0 \\ q_1 \\ \vdots \\ q_{N-2} \end{pmatrix}$$

In this notation, system (32) for $n = 1,2,...,N$ will take the form

$$A_N X_N = a_0 Q_N + a_1 Q_N' + a_2 Q_N'', \qquad (36)$$

where $X_N = (a_3\ a_4 \dots\ a_{N+1})^T$.

The solution of equation (36) is

$$X_N = A_N^{-1}(a_0 Q_N + a_2 Q_N'') + a_1 A_N^{-1} Q_N'. \qquad (37)$$

In these designations for $\varphi(k)$ we have the expression

$$\varphi(k) = 1 + a_1 k + a_2 k^2 + a_3 k^3 + \dots = 1 + a_1 k + a_2 k^2 + \lim_{N\to\infty} K_N X_N = 1 + a_1 k + a_2 k^2 + \lim_{N\to\infty} K_N A_N^{-1}\left(a_0 Q_N + a_2 Q_N''\right) + a_1 \lim_{N\to\infty} K_N A_N^{-1} Q_N', \qquad (38)$$

where $K_N = (k^3, k^4, \dots, k^{N+2})$.

Denote the two limits on the right-hand side of (38) by $\mathrm{h}(k)$ and $g(k)$ respectively. For a given $q(k)$ - these are known functions. Then (38) takes the form

$$\varphi(k) = 1 + a_1 k + a_2 k^2 + \mathrm{h}(k) + a_1 g(k) = a_1\big(k + g(k)\big) + 1 + a_2 k^2 + \mathrm{h}(k) \equiv a_1 g_1(k) + \mathrm{h}_1(k), \qquad (39)$$

where $k + g(k) = g_1(k), \quad 1 + a_2 k^2 + \mathrm{h}(k) = \mathrm{h}_1(k)$.

Relation (39) holds for $0 \le k < k_0$, where k_0 is big enough. The coefficient a_1 here is still undefined [8, 9]. To find it, we use the asymptotic solution $\varphi(k)$ ($k \to +\infty$), given by Theorem 1. Let $k_1 < k_0$. Then by Theorem 1

$$\begin{cases} \varphi(k) = C q^{-\frac{1}{4}}(k) e^{-\int_{k_1}^{k}\sqrt{q(t)}dt}\big(1 + \varepsilon_2(k)\big) \\ \varphi'(k) = -C q^{\frac{1}{4}}(k) e^{-\int_{k_1}^{k}\sqrt{q(t)}dt}\big(1 + \varepsilon_1(k)\big), \end{cases} \qquad (40)$$

where $\varepsilon_1(k) \to_{K\to\infty} 0$, $\varepsilon_2(k) \to_{K\to\infty} 0$ and

$$C = C(k_1) = C_0 e^{-\int_0^{k_1}\sqrt{q(t)}dt} \xrightarrow[k_1\to\infty]{} 0. \qquad (41)$$

since by virtue of lemma 1

$$Re\left(\sqrt{q(t)}\right)_1 \geq \sqrt{\frac{\delta}{2}} > 0.$$

Insofar as $0 \leq k \leq k_1 < k_0$ formulas (39) and (40) give the same function $\varphi(k)$, we get for $0 \leq k \leq k_1 < k_0$

$$\begin{cases} a_1 g_1(k) + \mathrm{h}_1(k) = C(k_1) q^{-\frac{1}{4}}(k) e^{-\int_{k_1}^{k} \sqrt{q(t)}dt} \left(1 + \varepsilon_2(k)\right) \\ a_1 g_1'(k) + \mathrm{h}_1'(k) = -C(k_1) q^{\frac{1}{4}}(k) e^{-\int_{k_1}^{k} \sqrt{q(t)}dt} \left(1 + \varepsilon_1(k)\right), \end{cases} \quad (42)$$

Putting in (42) $k = k_1$, will get

$$\begin{cases} a_1 g_1(k_1) + \mathrm{h}_1(k_1) = C q^{-\frac{1}{4}}(k_1)\left(1 + \varepsilon_2(k_1)\right) \\ a_1 g_1'(k_1) + \mathrm{h}_1'(k_1) = -C q^{\frac{1}{4}}(k_1)\left(1 + \varepsilon_1(k_1)\right). \end{cases} \quad (43)$$

If $k_1 \gg 1$, then $|\varepsilon_1(k_1)| \ll 1,\ |\varepsilon_2(k_1)| \ll 1.$

Therefore, (43) can be approximately replaced by the system

$$\begin{cases} \tilde{a}_1 g_1(k_1) + \mathrm{h}_1(k_1) = \tilde{C} q^{-\frac{1}{4}}(k_1) \\ \tilde{a}_1 g_1'(k_1) + \mathrm{h}_1'(k_1) = -\tilde{C} q^{\frac{1}{4}}(k_1), \end{cases} \quad (44)$$

where $\tilde{a}_1$ and $\tilde{C}$ - are approximate values for a_1 and C .

From (44) we find

$$\begin{cases} \tilde{a}_1 = -\frac{\mathrm{h}_1 q^{\frac{1}{2}} + \mathrm{h}_1'}{g_1 q^{\frac{1}{2}} + g_1'}, \\ \tilde{C} = q^{\frac{1}{4}} \frac{g_1' \mathrm{h}_1 - g_1 \mathrm{h}_1'}{g_1 q^{\frac{1}{2}} + g_1'}, \end{cases} \quad (45)$$

where all functions are calculated when $k = k_1$.

For an approximate value $\tilde{\varphi}(k)$ of $\varphi(k)$ we therefore have

$$\begin{cases} \tilde{\varphi}(k) = \begin{cases} \tilde{a}_1 g_1(k) + \mathrm{h}_1(k),\ \ 0 \leq k \leq k_1 \\ \left(\frac{g_1' \mathrm{h}_1 - g_1 \mathrm{h}_1'}{g_1 q^{\frac{1}{2}} + g_1'}\right)_{k=k_1} q^{\frac{1}{4}}(k_1) q^{-\frac{1}{4}}(k) e^{-\int_{k_1}^{k} \sqrt{q(t)}dt}, k \geq k_1 \\ k_1 \gg 1,\ k_1 < k_0 \end{cases} \\ \tilde{\varphi}(-k) = \tilde{\varphi}(k),\ k \geq 0 \end{cases} \quad (46)$$

Remarks. 1. Exact values a_1 and $C = C(k_1)$ can be found as follows. Supposing $q(k)$ is entire function, i.e. decomposition $q(k) = q_0 + q_1 k + q_2 k^2 + ..$ has a place for each$k \in (-\infty, +\infty)$. Then relations (42), (43) are valid for all k_1 and k. Then from (42) we have:

$$\begin{cases} a_1 = -\frac{\mathrm{h}_1(k) q^{\frac{1}{2}}(k)\left(1+\varepsilon_1(k)\right) + \mathrm{h}_1'(k)\left(1+\varepsilon_2(k)\right)}{g_1(k) q^{\frac{1}{2}}(k)\left(1+\varepsilon_1(k)\right) + g_1'(k)\left(1+\varepsilon_2(k)\right)}, \\ C(k_1) = q^{\frac{1}{4}}(k) e^{-\int_{k_1}^{k} \sqrt{q(t)}dt} \frac{g_1'(k)\mathrm{h}_1(k) - g_1(k)\mathrm{h}_1'(k)}{g_1(k) q^{\frac{1}{2}}(k)\left(1+\varepsilon_1(k)\right) + g_1'(k)\left(1+\varepsilon_2(k)\right)}. \end{cases} \quad (47)$$

Passing to the limit by $k \to \infty$ and considering that $\varepsilon_j \to 0,\ j = 1{,}2$, we'll get

$$\begin{cases} a_1 = -\lim\limits_{k\to+\infty} \frac{\mathrm{h}_1(k)q^{\frac{1}{2}}(k)+\mathrm{h}_1'(k)}{g_1(k)q^{\frac{1}{2}}(k)+g_1'(k)}, \\ C(k_1) = \lim\limits_{k\to+\infty} q^{\frac{1}{4}}(k)e^{-\int_{k_1}^{k}\sqrt{q(t)}dt}\frac{g_1'(k)\mathrm{h}_1(k)-g_1(k)\mathrm{h}_1'(k)}{g_1(k)q^{\frac{1}{2}}(k)+g_1'(k)}. \end{cases} \tag{48}$$

2. If in (38) we restrict ourselves to a finite number of terms, then for an approximate value of $\varphi_N(k)$, $0 \le k < k_0$ we will have

$$\varphi_N(k) = 1 + a_1k + a_2k^2 + K_NA_N^{-1}\left(a_0Q_N + a_2Q_N''\right) + a_1K_NA_N^{-1}Q_N' \equiv 1 + a_1k + a_2k^2 + \mathrm{h}_N(k) + a_1g_N(k) = a_1\left(k + g_N(k)\right) + 1 + a_2k^2 + \mathrm{h}_N(k) \equiv a_1g_{1,N}(k) + \mathrm{h}_{1,N}(k),\ \ 0 \le k \le k_0,\ k_0 \gg 1,\ N \gg 1 \tag{49}$$

where $g_{1,N}(k) = k + g_N(k)$, $\mathrm{h}_{1,N}(k) = 1 + a_2k^2 + \mathrm{h}_N(k)$.

Approximations $\tilde{a}_1{}^N$ и $\tilde{C}^N$ are found by formulas of the form (45), where h_1 and g_1 are replaced by $\mathrm{h}_{1,N}$ and$g_{1,N}$ respectively; $\tilde{\varphi}_N$ is determined similarly [8,9].

3. Matrix A_N^{-1} has the form:

$$A_N^{-1} = \begin{pmatrix} \frac{1}{2\cdot3} & 0 & 0 & 0 & 0 \\ b_{21} & \frac{1}{3\cdot4} & 0 & 0 & 0 \\ b_{31} & b_{32} & \frac{1}{4\cdot5} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{N1} & b_{N2} & \cdot & b_{N,N-1} & \frac{1}{(N+1)(N+2)} \end{pmatrix},$$

where b_{ks} $(k > s)$ are expressed through.q_i, $i \le k-3$; $k \ge 3$; $b_{21} = 0$ 4. Results

An algorithm for the analytical solution of the Kolmogorov-Feller equation (1) is constructed and substantiated. This algorithm is as follows.

1. Enter the function $\varphi(k) = \widehat{W}(k)e^{i\frac{\alpha}{2\beta}k}$, where $\widehat{W}(k)$– is a Fourier transform of the desired function $W(x)$.

2. For function $\varphi(k)$ the equation holds

$$\varphi''(k) - q(k)\varphi(k) = 0, \quad k > 0$$

under conditions [10]

$$\begin{cases} \varphi(0) = 1 \\ \varphi(k) \to 0, \quad k \to +\infty' \end{cases}$$

$$q(k) = q_0 + q_1k + q_2k^2 + \ldots, \quad |k| < k_0.$$

3. The coefficients q_j are from the relations:

$$\begin{cases} q_0 = -\frac{\alpha^2}{2\beta^2} + i\frac{\nu}{\beta}\hat{p}_1 \\ q_n = i\frac{\nu}{\beta}\hat{p}_{n+1} \end{cases},$$

where $\hat{p}_j$ – are from equalities $\hat{p}_s = \frac{\hat{p}^{(s)}(0)}{s!} = \frac{1}{s!}(i)^s\int_{-\infty}^{+\infty} x^s p(x)\ dx$, or as decomposition coefficients of the Fourier transform $\hat{p}(k) = \int_{-\infty}^{+\infty} p(x)\ e^{ixk}dx$ by powers of variable k:

$$\hat{p}(k) = \hat{p}_0 + \hat{p}_1k + \hat{p}_2k + \cdots, |k| < k_0, \qquad k_0 \gg 1$$

4. The sought solution $\varphi(k)$ has the appearance

$$\varphi(k) = 1 + a_1 k + a_2 k + \cdots, \quad 0 \le k < k_0$$

where the coefficients $a_j,\ j \ge 2$ are sequentially determined from a system of linear algebraic equations

$$(n+1)(n+2)a_{n+2} - \sum_{s=0}^{n} a_s q_{n-s} = 0$$

The determinant of this system is non-zero.

The coefficient a_1 is determined by the ratio

$$a_1 = -\lim_{k\to+\infty} \frac{\mathrm{h}_1(k) q^{\frac{1}{2}}(k) + \mathrm{h}_1'(k)}{g_1(k) q^{\frac{1}{2}}(k) + g_1'(k)},$$

where the functions $\mathrm{h}_1(k),\ g_1(k)$ are determined by the ratios (38), (39).

Conclusions

The results presented in this article were obtained as a result of investigation. The main objective of the investigation was the development and theoretical substantiation of a method for constructing algorithms for the analytical and numerical solution. Besides, the aim was to develop the analysis of transfer and filtration problems based on special classes of differential, integral and integro-differential equations, appropriate to applied problems. These include the problems of system analysis, the theory of random processes, the theory of automatic control, as well as the analytical solution of a number of new problems. The new problems are the construction and study of solutions of the Kolmogorov-Feller equation with a quadratic drift coefficient, solutions of the Boltzmann equation of the kinetic theory of gases, and the problem of the Poisson processes filtration.

References

1. Tikhonov V.I., Mironov M.A. Markov processes. Moscow, Soviet radio, 1977. (in Russian)
2. Tikhonov V.I., Kharisov V.N. Statistic analysis and synthesis of radio devices and systems. Moscow, Radio and communication, 1991. (in Russian)
3. Aleksandrov V.D. Exact solution of the stationary Kolmogorov-Feller equation // Review of applied and industrial mathematics. 2002. vol. 9. i. 1. P. 106, (in Russian)
4. Artemyev V.M., Ivanovskyi A.V. Discrete control systems with a random quantization period. Moscow, Energoatomizdat, 1986. 458 p. (in Russian)
5. Fedoryuk M.V. Ordinary differential equations. Moscow, Nauka, 1980. 458 p. (in Russian)
6. Rudenko O.V., Dubkov A.A., Gurbatov S.N. On exact solutions of the Kolmogorov–Feller equation // Doklady Akademii nauk. 2016.vol. 469 (4). P.414-418.
7. King I.R. An Introduction to Classical Stellar Dynamics. Berkeley: University of California, 1994.
8. Zhilenkov A., Chernyi S., Sokolov S., Nyrkov A., Intelligent autonomous navigation system for UAV in randomly changing environmental conditions // J. Intell. Fuzzy Syst. 2020. 1–7. doi:10.3233/jifs-179741
9. Sokolov S., Zhilenkov, A., Chernyi, S., Nyrkov, A., & Mamunts, D. Dynamics Models of Synchronized Piecewise Linear Discrete Chaotic Systems of High Order // Symmetry. 2019. 11(2). 236. doi: 10.3390/sym11020236
10. Zhilenkov A.A., Chernyi S.G. Automatic estimation of defects in composite structures as disturbances based on machine learning classifiers oriented mathematical models with uncertainties // Journal of Information Technologies and Computing Systems. 2020. № 3. C. 13-29.

Zhilenkov A. A. St. Petersburg State Marine Technical University (SMTU), St. Petersburg, Russian Federation Head of the Department of Cyberphysical Systems, Cand. Sci. (Eng.), Assoc. Prof. Publications: 167 (3) Research interests: cyber-physical systems, automatic control, modeling, artificial intelligence, robotics. Email: zhilenkovanton@gmail.com

Chernyi S. G. Kerch State Maritime Technological University, 2983095, Russia, Kerch, Ordzonikidze st., 82. Head of the Department of Ship Electrical Equipment and Industrial Automation, PhD. Admiral Makarov State University of Maritime and Inland Shipping, 198035, Russia, Saint-Petersburg, Dvinskaya st., 5/7. Associate Professor, PhD, Associate Professor. Number of publications: 100 (including 3 monographs and 2 books). IEEE, DAAAM, EAI Membership. Member of the Editorial Board of MDPI, Hindawi journals. Research interests: information technology, modeling of heterogeneous systems. Email:sergiiblack@gmail.com

Firsov A. N. St. Petersburg Polytechnic University, St. Petersburg, Russian Federation, Dr., Professor - Higher School of Cyber-Physical Systems and Control. Publications: 167 (3) Research interests: cyber-physical systems, modeling. Email: andrey_firsov@spbstu.ru